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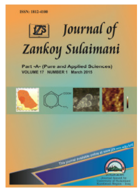

Subdividing Extension of Eulerian Graphs

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Abstract

In this paper, the subdividing extensibility number for Eulerian graphs has been studied. Eulerian graphs with subdividing extensibility number at most 3 are characterized.

I- Introduction

Attar [5] introduced the concept of vertex extension of graphs (digraphs) and the definition of extensibility number for graphs (digraphs). Further, he characterized the vertex extensibility number for some graphs (digraph). Attar [2] also introduced the edge extension of graphs (digraphs), and he introduced the concept of edge extension class of graphs, and established the edge extensibility number of some graphs (digraphs), Attar and Ahmid [4] improved the definition of extension graphs (digraphs), and they characterized the regular graphs (digraphs) with extensibility number 1, 2 or 3. Furthermore, Attar and Ahmed [3] studied the extensibility number for eulerian graphs (digraph) and they established the eulerian graphs which have extensibility number at most 3. In this work we use the subdividing of edges to extend the Eulerian graph.

II. Subdividing Extension of Graphs

Here, we define the extension of G by subdividing edge.

Definition 1: Let G be a simple graph with $n \geq 3$ vertices and at least one edge. The extension of G by subdividing edge $e \in E(G)$ is a simple graph, G_e obtained from G by subdividing an edge e = uv of G by the vertex x such that x is adjacent to at least one vertex of G different from the end vertices of e.

In [1] Akram introduced the following definsions:

Definition 2: Let G be a nonnull simple graph with, $n \geq 3$ vertices. Given a set of edges $S = \{e_1, e_2, \dots, e_m\}$ of G. We can construct a new simple graph from G by an extension subdividing the edges of S by a new set of vertices $X = \{x_1, x_2, \dots, x_m\}$ respectively, in such a way that every vertex of X is adjacent to at least one vertex of G different from its neighbours. We denote the new graph by G_S. We say that G_S is obtained from G by subdividing extension edges and S is called subdividing extension set. In particular, if S consists of a single element e, then e is called subdividing extension edge and the graph denoted by G_e . The transition from G to G_S is called subdividing extension operation.

Definition 3: let $\mathfrak{R}$ be a class of graphs with certain property. Then $\mathfrak{R}$ is called subdividing extensible if for every graph $G \in \mathfrak{R}$, either G is a null graph or there exist a subdividing extension edge e in G such that $G_e \in \mathfrak{R}$.

Example1: The class of bipartite graphs is not subdividing extensible.

Definition 4: Let ξ be a class of graphs with certain property, and G $\in \xi$ be a non trivial(nonnull) with $n \geq 3$. The subdividing extensibility number of G with respect to ξ is the smallest positive integer m, if exists, such that there exists a subdividing extension set S of cardinality m in G in such away the graph $G_S \in \xi$. We write $m = sub - ext_{\xi}(G)$. If such a number does not exist for G, then we say that the corresponding subdividing extensibility number of G is ∞.

In this work, we studied the subdividing extensibility number for eulerian graphs.

We need the following theorem regarding characterization of eulerian graphs.

Theorem 1 [8]: The connected graph G is eulerian if and only if the degree of every vertex of G is even.

For the undefined concepts of graphs, we refer the reader to [6] and [8]. All the graphs throughout this paper are simple connected graphs.

Proposition 1: Let $\mathfrak{I}$ be the class of eulerian graphs, if $G \in \mathfrak{I}$, then $sub - ext_{\mathfrak{I}}(G) > 1$.

Proof: Let $\mathfrak{I}$ be the class of eulerian graphs, and let $G \in \mathfrak{I}$. Then by Theorem 1, every vertex of G has an even degree. Let $e = uv$ be an edge in G subdivided by the vertex x. Suppose that e is subdividing extension edge of G, then x is adjacent to at least one vertex of G other than the end vertices of e. In this case the degree of every vertex which is adjacent by x is odd. That is $G_e \notin \mathfrak{I}$. Hence $sub - ext_{\mathfrak{I}}(G) > 1$. □

Theorem 2: Let $\mathfrak{I}$ be the class of eulerian graphs, and let $G \in \mathfrak{I}$. Then $sub - ext_{\mathfrak{I}}(G) = 2$ if and only if there exist two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ of G which can subdivided by a new vertices x and y respectively and the following properties are holds:

1. Each of x and y is adjacent to even number of vertices in $G - \{u_1, v_1, u_2, v_2\}$,
2. $N(x) - \{u_1, v_1\} = N(y) - \{u_2, v_2\}$, and
3. Every vertex of $\{u_1, v_1, u_2, v_2\}$ is either a common end vertex of x, y or is a neighbor to exactly one vertex of x, y.

Proof: Let $\mathfrak{I}$ be the class of eulerian graphs, and let $G \in \mathfrak{I}$. Suppose that $sub - ext_{\mathfrak{I}}(G) = 2$. Then by Definition 4 , there exist subdividing extension set $S = \{e_1, e_2\}$ of cardinality 2 such that $G_s \in \mathfrak{I}$, and S is the smallest such set. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$, and let x, y be the two vertices which come from subdividing the edges e_1 and e_2 respectively. .As e_1, e_2 are edges in the simple graph G, then the induced subgraph $< e_1, e_2 >$ in G is isomorphic to one of the following two subgraphs: $P_2 \cup P_2$ or P_3.

As G_S is eulerian, then by Theorem 1, every vertex of G_S has even degree. If x is adjacent to odd number of vertices of $G - \{u_1, v_1\}$, then x has even degree in G_S a contradiction, similarly for y and 1 holds.

Suppose that the induced subgraph which induced by $S = \{e_1, e_2\}$ is isomorphic to $P_2 \cup P_2$. As G_S eulerian, then every vertex of $G - \{u_1, v_1, u_2, v_2\}$ which is adjacent by x is adjacent by y. That is $N(x) - \{u_1, v_1\}$ $= N(y) - \{u_2, v_2\}$ and 2 holds. Since u_1 is already adjacent by x, then if u_1 is adjacent by y, then the degree of u_1 in G_S is odd a contradiction. Similarly for v_1, u_2 *and* v_2. Hence every vertex of $\{u_1, v_1, u_2, v_2\}$ is adjacent by exactly one vertex of x, y and 3 holds. Now, suppose that the induced subgraph which induced by e_1, e_2 is isomorphic to P_2. In this case e_1, e_2 have a common end vertex say $v_1 = u_2$. By similar argument to that in the first case above, every vertex which is adjacent by x in $G - \{u_1, v_1, u_2, v_2\}$ must be adjacent by y and $N(x) - \{u_1, v_1\} = N(y) - \{u_2, v_2\}$ in $G - \{u_1, v_1, u_2, v_2\}$. Since u_1 is a neighbor of x, then if u_1 is adjacent by y then u_1 has odd degree in G_S a contradiction to the definition of eulerian graph, similarly for v_2. Then each of the vertices u_1, v_2 is adjacent by exactly one vertex from x, y.

Conversely, suppose that G contain two edges $e_1 = u_1v_1$, $e_2 = u_2v_2$ are subdivided by the vertices x, y respectively and the conditions 1,2 and 3 above are holds. Then each of x, y has even degree in G_S, and every vertex in $G-\{u_1,v_1,u_2,v_2\}$ which is adjacent by x is adjacent by y, that is its degree increase by 2, but his degree is even. Then the degree of every vertex in $G-\{u_1,v_1,u_2,v_2\}$ which is adjacent by x and y is even in G_S. From condition 3 each vertex from $\{u_1,v_1,u_2,v_2\}$ has even degree in G_S. Then every vertex in G_S has even degree. That is G_S is eulerian and $S=\{e_1,e_2\}$ is subdividing extension set. Thus $sub-ext_{\Im}(G)\le 2$. By Proposition 1, $sub-ext_{\Im}(G)>1$. Hence $sub-ext_{\Im}(G)=2$.

Theorem 3: Let $\Im$ be the class of eulerian graphs, and let $G\in\Im$ with n vertices. Then $sub-ext_{\Im}(G)=3$ if and only if G contain three edges e_1,e_2,e_3, which can subdivided by the vertices x,y,z respectively, and the followings properties holds.

(1) Each of x,y,z is adjacent to even number of vertices in G other than the end vertices of the subdivided edge.

(2) Each vertex in $G-\{u_1,v_1,u_2,v_2,u_3,v_3\}$ which is adjacent by some vertex of x,y,z is a neighbor to exactly two vertices of x,y,z.

(3) Each end vertex of $\{u_1,v_1,u_2,v_2,u_3,v_3\}$ which is not a common end vertex of exactly two vertices from x,y,z and adjacent by two vertices of x,y,z is a common neighbor for x,y,z.

Proof: Let $\Im$ be the class of eulerian graphs, $G\in\Im$. Suppose that $sub-ext_{\Im}(G)=3$. Then by Definition 4, there exists a subdividing extension set $S=\{e_1,e_2,e_3\}$ of cardinality 3 such that $G_S\in\Im$ and S is the smallest such set. Let $e_1=u_1v_1, e_2=u_2v_2, e_3=u_3v_3$ be the edges of S which subdivided by x,y,z respectively.

As G is eulerian, then by Theorem 1, the degree of every vertex in G is even. Since G_S is simple eulerian, then the degree of every vertex of x,y,z is even. That is each of x,y,z must be adjacent to even number of vertices in G other than the end vertices of its subdivided edge and (1) holds. As we have a set of three edges $S=\{e_1,e_2,e_3\}$. Then the induced subgraph which induced by $[S]$ in G is isomorphic to one of the following graphs: $P_2\cup P_2\cup P_2, C_3, P_3\cup P_2, P_4, S_4$.

In each of the 5th subgraphs. Suppose that h is a vertex in $G-\{u_1,v_1,u_2,v_2,u_3,v_3\}$ which is adjacent by 1 or 3 vertices from the vertices x,y,z. In this case the degree of h is odd in G_S a contradiction. Thus every vertex in $G-\{u_1,v_1,u_2,v_2,u_3,v_3\}$ which is adjacent by some vertex of x,y,z is adjacent by exactly two vertices from x,y,z and (2) holds.

Now, in each of the 5th subgraphs. Suppose that the end vertex u_1 of e_1 is not a common end vertex of exactly two edges of e_1,e_2,e_3 and u_1 is a neighbor of two vertices from x,y,z, in this case u_1 has odd degree in G_S which is a contradiction. Hence u_1 must be a common neighbor for x,y,z and (3) holds.

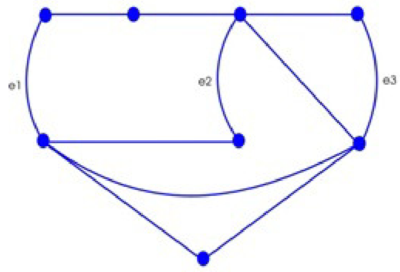
Conversely, suppose that G contain three edges e_1,e_2,e_3 which can subdivided by the vertices x,y,z respectively and the conditions (1), (2) and (3) above are holds.

Then each of x,y,z has even degree in G_S and every vertex in $G-\{u_1,v_1,u_2,v_2,u_3,v_3\}$ which is adjacent by some vertex of x,y,z has even degree in G_S and every end vertex of $\{u_1,v_1,u_2,v_2,u_3,v_3\}$ has even degree in G_S. Then every vertex in G_S has even degree, that is G_S is eulerian and $S=\{e_1,e_2,e_3\}$ is subdividing extension set. Hence $sub-ext_{\Im}(G)\le 3$, by Proposition 1, $sub-ext_{\Im}(G)>1$. Suppose that $sub-ext_{\Im}(G)=2$, then there exist two edges $e_1=uv$, $e_2=wh$ of G can subdivided by a new vertices $x_\circ, y_\circ$ respectively and the following holds:

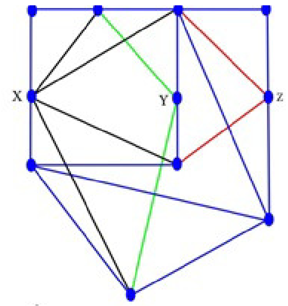
1. each of $x_\circ, y_\circ$ is adjacent to even number of vertices in $G-\{u,v,w,h\}$, and
2. $N(x_\circ)-\{u,v\}=N(y_\circ)-\{w,h\}$, and
3. every vertex of $\{u,v,w,h\}$ is either a common end vertex of $x_\circ, y_\circ$ or is a neighbor to exactly one vertex of $x_\circ, y_\circ$, a contradiction to our assumption. Thus $sub-ext_{\Im}(G)\neq 2$. Hence $sub-ext_{\Im}(G)=3$.

To illustrate Theorem 3, we introduce the following example:

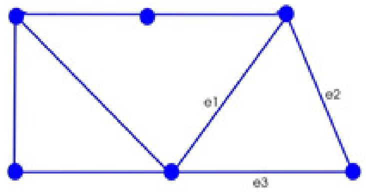
Example 2 : Consider the graphs $G_1,G_2,G_3,\dots,G_5$ depicted in Figure 1. From each of the given graphs, we subdivided three edges e_1,e_2,e_3 by the new vertices x,y,z respectively. We see that the induced subgraphs $<e_1,e_2,e_3>$ is isomorphic to one of the graphs $P_2\cup P_2\cup P_2, C_3, P_3\cup P_2, P_4, S_4$ respectively.



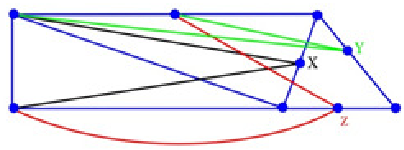
G1



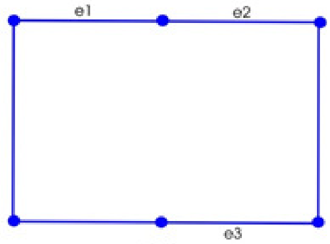
G_1^- subdividing the edges $e_1 e_2 e_3$ of G by the vertices x, y, z resp., where e_1, e_2 and e_3 are independent edges.



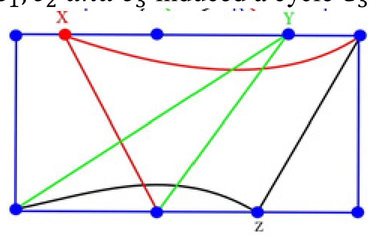
G2



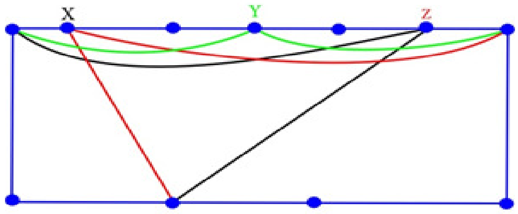
G_2^- subdividing the edges $e_1 e_2 e_3$ of G by the vertices x, y, z resp., where e_1, e_2 and e_3 induced a cycle C_3.



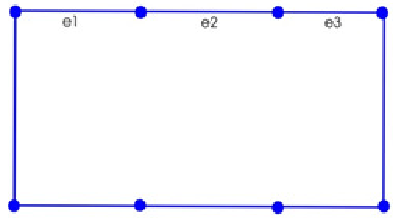
G3



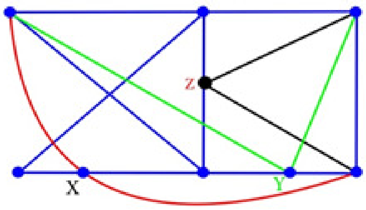
G_3^- subdividing the edges $e_1 e_2 e_3$ of G by the vertices x, y, z resp., where e_1 and e_2 are adjacent while e_3 is not adjacent to any of them.



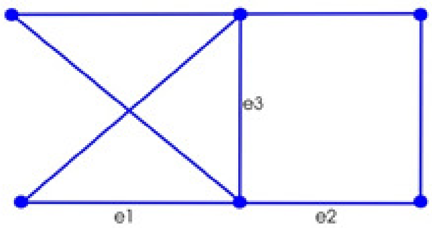
G_4^- subdividing the edges $e_1 e_2 e_3$ of G by the vertices x, y, z resp., where e_1, e_2 and e_3 induced a path p_4.



G4



G_5^- subdividing the edges $e_1 e_2 e_3$ of G by the vertices x, y, z resp., where e_1, e_2 and e_3 induced a star graph.



G5

Figure 1. Classes of Eulerian Graphs with $m = 3$

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